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HYDROELASTIC VIBRATIONS IN A RECTANGULAR CONTAINER

HELMUT F. BAUER
Hochschule der Bundeswehr München, Fachbereich Luft- und Raumfahrttechnik, W. Germany

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Abstract—The interaction of an elastic bottom with the liquid exhibiting a free liquid surface has been investigated for a rectangular container. For this reason the container bottom was considered either as a flexible membrane or as a thin elastic rectangular plate. Furthermore the hydroelastic problem of a liquid in a rigid rectangular tank in which the free liquid surface was covered by a flexible membrane or a thin elastic plate has also been treated. In both cases the coupled frequencies of the structure-liquid system has been obtained. It was found that even structural modes couple with odd liquid modes and vice versa and that the coupled frequencies exhibit decreased magnitude compared with the uncoupled structural frequencies and increased magnitude compared to the uncoupled liquid frequencies. They decrease with decreasing tension of the membrane or decreasing stiffness of the plate.

NOTATION

$\Phi(x, y, z, t)$	velocity potential
a	width of the container
b	depth of the container
h	liquid height in the container
$w(x, y, t)$	displacement of membrane or plate
ρ	density of liquid
T_0	tension in membrane
g	gravitational or longitudinal acceleration
μ	mass/unit area of membrane
T	kinetic energy
V	potential energy
ρ_p	mass density of the plate
h_p	thickness of the plate
D	stiffness of the plate
$P(x, y, t)$	loadfunction
ν	Poisson ratio
$\omega_{mn}^{(s)}$	uncoupled liquid slosh frequencies
$\omega_{mn}^{(M)}$	uncoupled membrane frequencies
$\omega_{mn}^{(P)}$	uncoupled plate frequencies
ω_{mn}	coupled frequencies

1. INTRODUCTION

Liquid containers of present and future aerospace vehicles are by no means rigid. The motion of a liquid with a free surface is induced by rigid body or by elastic wall excitation and may have severe influence upon the stability of the vehicle. The low fundamental frequency of the liquid, which is usually very close to the control frequency of the vehicle, affects the coupling of control, of liquid and the elastic walls considerably, since with the increasing size of the vehicles their elastic frequencies become significantly lower. Thus, the problem of interaction has a pronounced influence upon the design of such a vehicle and its control system. Many investigators have studied the sloshing of liquid with a free surface in order to determine the natural frequencies and the response of the liquid to various excitations. For these studies the containers of various geometries have been considered to be perfectly rigid, while the liquid could be treated as inviscid and incompressible. For simple container geometries, such as circular annular cylindrical sector tanks[1, 2] or rectangular containers[3, 4] the procedure for the solution of the linearized liquid motion is rather straightforward and yields very useful results for the interaction of liquid motion and vehicle control. For a more complicated container geometry, such as conical, spherical or composite construction of cylindrical shells with some kind of hemispherical or elliptical bulkhead, the determination of the behavior of the liquid with a free surface becomes even in linearized theory quite difficult and may be treated by approximate numerical methods only. For additional information on this subject, Abramson[5] may be consulted. In large aerospace vehicles the propellant containers increase in size and decrease in stiffness and the coupled frequencies of the liquid-structure-system may be significantly different from those of the uncoupled system. They may be even close to the control frequency and may therefore be of distinct danger to the performance and mission of the aerospace vehicle. The immediate extension of the earlier work on liquid sloshing in rigid containers is the case including the response of the liquid due to the motion of the elastic structure [6, 7]. In such a case the treatment is focused on the behavior of the liquid with a free surface in a container whose walls were forced to oscillate with a prescribed shape and frequency. The analysis gives, however, no allowance for the interaction of liquid and motion of the elastic structure and it is justified only as long as the dominant natural frequencies of the liquid and the structure are well apart from each other, and as long as the generalized masses are not too large. Further work in this direction has been performed by treating the elastic portion of the containers as membranes [8–10]. A more general analysis of the complete coupled bending and sloshing was made by Rabinovich [11] and Miles [12]. In these investigations potential flow was assumed and a velocity potential could be determined. Miles' analysis of the coupled bending-sloshing problem involves the use of the Lagrangian procedure. The result of the analysis exhibited a lowering of the resonant bending frequencies due to the sloshing liquid in the container, while the coupled sloshing frequencies did not appreciably change from their uncoupled values. Lindholm [13] *et al.* conducted experiments in which the couple effect on the bending frequency was measured for thin cylindrical shells. It was found that the theoretical results of Miles give a fairly adequate prediction of the influence of the liquid motion upon the bending frequency of the cylindrical container. Breathing vibrations of a partially filled cylindrical container have been investigated by Chu [14] and Chu and Gonzales [15]. They considered shell modes that display both circumferential and axial wave patterns for rotationally symmetric containers, for which they neglected circumferential and longitudinal inertia. It was found that for shell frequencies higher than those of the first several uncoupled liquid modes the surface effects on the breathing motion of the shell are of negligible magnitude unless the excitation is in the vicinity of a low sloshing frequency. Further work involving coupled oscillations of liquid and elastic containers is limited and in some cases inconclusive and not in good agreement with experimental results. Fontenot and Lianis [16] have investigated a completely filled cylindrical shell by using a perturbation technique, while Rabinovich [17] treated a partially filled cylindrical shell employing the Mushtari–Donnell–Vlasov shell equations. The results of a full container compare quite well with the experimental data [18]. For a container with elastic walls and bottom, Bauer [7] presented a procedure for the determination of the coupled frequencies for axisymmetric oscillations. For the cylindrical shell he used Donnell's shell equations, while in the case of the container bottom a flexible membrane or an elastic plate is used. Natyshkin and Rakhimov [19] have also performed some investigations of a partially filled cylindrical container with various end conditions. Systems being partially elastic and rigid have been treated by various authors. Bhuta and Koval have investigated the interaction between the liquid surface oscillation in a cylindrical container having a rigid wall and a thin flat membrane [20] or plate bottom [21].

The coupled frequencies were found to be slightly lower than those for a completely rigid container. They exhibit only a marked difference for low fillings. More on the hydroelastic behavior of a system is presented for circular cylindrical containers in [22] and [23] and for axisymmetric oscillations in an annular cylindrical tank in [24]. Bauer *et al.* [25] treated the axisymmetric case of a container with elastic side-walls and rigid bottom. In this paper they also investigated the coupled motion of a liquid in an infinitely large rectangular container with elastic sidewalls and rigid bottom and elastic bottom and rigid sidewalls. It was found that with decreasing ratio of sidewall height to liquid height the fundamental coupled frequency increases considerably for elastic sidewalls and exhibits quite different values to the uncoupled frequencies. For rigid sidewalls and an elastic container bottom the coupled frequency is always smaller than that of the liquid in the completely rigid container. The case of a cylindrical tank with both sidewall and bottom being elastic has been treated in [26], while nonlinear liquid motion in a longitudinally excited container with an elastic plate bottom was treated by Bauer *et al.* [27]. Other hydroelastic oscillations in a rigid circular cylindrical container with coverage of the free liquid surface by a flexible membrane or an elastic plate has been treated by Bauer [28] for free and forced oscillations of the container. This way could the liquid frequencies through the coupling with the elastic lid be shifted to much higher values, thus exhibiting the benefit of a more effective separation from a control frequency.

For rectangular containers such hydroelastic investigations have not been performed, inspite of the fact that they play an important role in aircraft, space craft and ship design. For such a geometry the analysis is due to the appearing coordinates analytically more involved since the method of separation fails. The present paper is dealing therefore with the formulation and solution of the hydroelastic problem of a rectangular container partially filled with a nonviscous and incompressible liquid. The container shall have rigid sidewalls and an elastic bottom in form of a flexible membrane or an elastic plate or it shall have a rigid tank-bottom and a liquid surface being covered by a membrane or a plate. In both cases the coupled frequencies of the system shall be obtained. This is useful information not only for the design of spacecraft and aircraft vehicles, but also for building of ships, especially large tankers, as well as for structural systems that have to be designed to withstand earthquakes.

2. BASIC EQUATIONS

A rectangular container with length a and width b is filled to a height h with a homogeneous, nonviscous and incompressible liquid. It may have an elastic container-bottom and a free liquid surface or its liquid surface may be covered by an elastic membrane or plate. If the displacement of the liquid and the elastic structure is considered small, the interaction of the structure and liquid may be treated with their linearized equations of motion. Since the flow of the liquid in the container may be considered irrotational, the velocity of the liquid can then be represented as the gradient of a velocity potential Φ. From the continuity equation div $\Phi = 0$ we obtain thus the Laplace equation (Fig. 1)

$$\Delta\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0 \tag{1}$$

which has to be satisfied in the region

$$0 \le x \le a, \quad 0 \le y \le b, \quad -h \le z \le 0.$$

The boundary conditions at the rigid sidewalls are given for the stationary container by

$$\frac{\partial\Phi}{\partial x} = 0 \quad \text{at the side walls} \quad x = 0, a \text{ and} \begin{Bmatrix} -h \le z \le 0 \\ 0 \le y \le b \end{Bmatrix} \tag{2}$$

and

$$\frac{\partial\Phi}{\partial y} = 0 \quad \text{at the side walls} \quad y = 0, b \text{ and} \begin{Bmatrix} -h \le z \le 0 \\ 0 \le x \le a \end{Bmatrix}. \tag{3}$$

If the container is excited harmonically in x-direction by $x(t) = x_0 e^{i\Omega t}$ with the forcing frequency Ω the sidewall boundary condition (2) yields then

$$\frac{\partial\Phi}{\partial x} = x_0 i\Omega\, e^{i\Omega t} \quad \text{at} \quad x = 0, a. \tag{4}$$

In case of a rotational excitation $\vartheta(t) = \vartheta_0 e^{i\Omega t}$ about an axis through the undisturbed center of

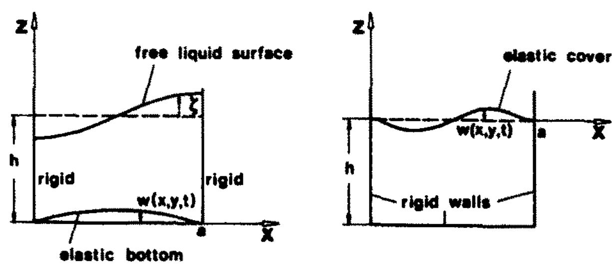

Fig. 1. Coordinates of the hydroelastic systems.

gravity (new coordinate system) the sidewall boundary condition (2) is to be replaced by

$$\frac{\partial \Phi}{\partial x} = \vartheta_0 i\Omega z\, e^{i\Omega t} \quad \text{at} \quad x = \pm\frac{a}{2}. \tag{5}$$

If the rotational excitation $\varphi(t) = \varphi_0 e^{i\Omega t}$ takes place about an axis through the center of gravity of the liquid perpendicular to the free liquid surface (i.e. z-axis), then the wall-boundary conditions read

$$\frac{\partial \Phi}{\partial x} = i\Omega\varphi_0 e^{i\Omega t} y \quad \text{at} \quad x = \pm\frac{a}{2} \tag{6}$$

$$\frac{\partial \Phi}{\partial y} = i\Omega\varphi_0 e^{i\Omega t} x \quad \text{at} \quad y = \pm\frac{b}{2}. \tag{7}$$

2.1 *Free liquid surface and elastic bottom*

If the liquid in the container is free to oscillate the free surface condition is obtained from the kinematic condition $\Phi_z = \zeta_t$, expressing with ζ as the free liquid surface displacement the fact, that the normal velocity of the free surface is equal to the normal velocity of a fluid particle at the free liquid surface, and the unsteady Bernoulli equation, which yields in linearized form the expression $\Phi_t + g\zeta = 0$. The free liquid surface condition is therefore

$$\frac{\partial^2 \Phi}{\partial t^2} + g\frac{\partial \Phi}{\partial z} = 0 \quad \text{at the surface } z = 0 \left(\text{or } \frac{h}{2} \text{ in the case of rotation } \vartheta(t)\right). \tag{8}$$

Since the container bottom is considered to be elastic, its motion may be described by a flexible membrane or an elastic plate. For a *flexible clamped membrane* the equation of the motion of the membrane is given by

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} - \frac{\mu}{T_0}\frac{\partial^2 w}{\partial t^2} = -\frac{\rho}{T_0}\frac{\partial \Phi}{\partial t}\bigg|_{z=-h} - \frac{\rho g}{T_0} w \tag{9}$$

with the boundary condition

$$w = 0 \quad \text{for} \quad x = 0, a \quad \text{and} \quad y = 0, b. \tag{10}$$

Here w is the displacement of the membrane in z-direction, μ its mass per unit area, T_0 its tension, while ρ is the mass density of the liquid. Instead of the membrane equation (9), we could also use the kinetic and potential energy of the membrane and the loading $P(x, y, t)$ and derive the equation of motion with the help of the Lagrange equation and the virtual work of the loading of the membrane, in order to obtain the generalized force on it. The kinetic energy is given by

$$T = \frac{\mu}{2}\int_0^a \int_0^b \left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}x\, \mathrm{d}y. \tag{11}$$

The potential energy is

$$V = \frac{T_0}{2}\int_0^a \int_0^b \left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] \mathrm{d}x\, \mathrm{d}y \tag{12}$$

and the loading of the membrane by the liquid yields

$$P(x, y, t) = -\rho\frac{\partial \Phi}{\partial t}\bigg|_{z=-h} - \rho g w. \tag{13}$$

If the container bottom is described by the motion of an *elastic plate*, eqn (9) must be substituted by

$$D\Delta^2 w + \rho_p \bar{h}_p \frac{\partial^2 w}{\partial t^2} = -\rho \frac{\partial \Phi}{\partial t}\Big|_{z=-h} - \rho g w \tag{14}$$

where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Laplacian operator. For a simply supported plate the boundary conditions of the plate are given by

$$w = 0 \quad \text{at} \quad x = 0, a \quad \text{and} \quad y = 0, b \tag{15}$$

and

$$\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} = 0 \quad \text{at} \quad x = 0, a \tag{16}$$

$$\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} = 0 \quad \text{at} \quad y = 0, b \tag{17}$$

where ν is the Poisson ratio and $D = E\bar{h}_p^3/12(1-\nu^2)$ represents the flexural rigidity of the plate. E is Young's modulus of elasticity, $\bar{h}_p$ is the thickness of the plate and $\rho_p \bar{h}_p$ is the mass density per unit area of the plate. If one applies the kinetic and potential energy for the derivation of the equations of motion of the plate one uses

$$T = \frac{1}{2} \rho_p \bar{h}_p \int_0^a \int_0^b \left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}x\, \mathrm{d}y \tag{18}$$

and

$$V = \frac{D_p}{2} \int_0^a \int_0^b \left\{\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right]^2 - 2(1-\nu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right]\right\} \mathrm{d}x\, \mathrm{d}y. \tag{19}$$

The compatibility condition at the elastic bottom is

$$\frac{\partial \Phi}{\partial z} = \frac{\partial w}{\partial t} \quad \text{at} \quad z = -h. \tag{20}$$

2.2 *Coverage of the liquid surface with an elastic member*

If the liquid surface is covered by a flexible membrane, one is able to shift the natural frequency of the coupled liquid-structure system to much higher values. The equation of the membrane is then

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} - \frac{\mu}{T_0}\frac{\partial^2 w}{\partial t^2} = \frac{\rho}{T_0}\frac{\partial \Phi}{\partial t}\Big|_{z=0} + \frac{\rho g}{T_0} w \tag{21}$$

the equation of a covering plate is

$$D_p \Delta^2 w + \rho_p \bar{h}_p \frac{\partial^2 w}{\partial t^2} = -\rho \frac{\partial \Phi}{\partial t}\Big|_{z=0} - \rho g w. \tag{22}$$

The bottom of the container has to satisfy

$$\frac{\partial \Phi}{\partial z} = 0 \quad \text{at} \quad z = -h \tag{23}$$

if it is considered rigid, while the compatibility condition in this case is given by

$$\frac{\partial \Phi}{\partial z} = \frac{\partial w}{\partial t} \quad \text{at} \quad z = 0. \tag{24}$$

These equations have to be solved to obtain the response of the hydroelastic system, i.e. the coupled frequencies of the liquid structure system and its coupled vibrational behavior. One may recognize that for $D_p \to 0(T_0 \to 0), \rho_p \to 0(\mu \to 0)$ eqns (21) and (22) yield the dynamic condition of the free liquid surface ($w \to \zeta$), which together with the compatibility condition (kinematic condition) results in the free liquid surface condition (8).

3. METHOD OF SOLUTION

We shall treat first the coupled liquid oscillations in a container with a free liquid surface and an elastic bottom.

3.1 *Free liquid surface and elastic bottom*

To determine the coupled frequencies of a liquid with a free surface and an elastic bottom, we have to solve simultaneously eqns (1)–(3), (8) together with (9), (10) or (14)–(17) with (20).

We treat first the case of a flexible *membrane* as the container bottom. A velocity potential satisfying the Laplace equation (1) together with the tank sidewall boundary conditions (2) and (3) as well as the free surface condition (8) is given by

$$\Phi(x, y, z, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn} \, e^{i\omega t} \left\{ \cosh\left[\frac{\pi z}{ab}\sqrt{n^2a^2 + m^2b^2}\right] + \left(\frac{\omega^2 \tanh\left[\frac{\pi h}{ab}\sqrt{n^2a^2+m^2b^2}\right] - \frac{g\pi}{ab}\sqrt{n^2a^2+m^2b^2}}{(\omega_{mn}^{2(s)} - \omega^2)} \right) \sinh\left[\frac{\pi z}{ab}\sqrt{n^2a^2+m^2b^2}\right]\right\} \cdot \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right). \tag{25}$$

Here the free surface is taken at $z = h$, while the elastic bottom is at $z = 0$. The value $\omega_{mn}^{(s)}$ is the natural frequency of the liquid in a completely rigid container. It is [1]

$$\omega_{mn}^{2(s)} = \frac{g\pi}{ab}\sqrt{n^2a^2+m^2b^2}\tanh\left(\frac{\pi h}{ab}\sqrt{n^2a^2+m^2b^2}\right). \tag{26}$$

The A_{mm} are integration constants and ω are the coupled frequencies of the system, which have to be determined. The solution of the membrane satisfying the boundary conditions (10) is given by

$$w(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{mn}(t) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \tag{27}$$

where the time functions $W_{mn}(t)$ have to be determined by the membrane equation. The kinetic energy (11) yields with eqn (27)

$$T = \frac{\mu ab}{8} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \dot{W}_{mn}^2 \tag{28}$$

and the potential energy is given by

$$V = \frac{T_0 \pi^2 ab}{8} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\frac{m^2}{a^2} + \frac{n^2}{b^2}\right] W_{mn}^2. \tag{29}$$

With the loading of the membrane by the liquid motion $P(x, y, t)$ we are able to determine the generalized force Q_{mn} with the help of the virtual work

$$\delta W = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} Q_{mn}\,\delta W_{mn} = \int_0^a\int_0^b P(x, y, t)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\delta W_{mn}\sin\left(\frac{m\pi x}{a}\right)\cdot\sin\left(\frac{n\pi y}{b}\right)\mathrm{d}x\,\mathrm{d}y.$$

Introducing $P(x, y, t)$ (eqn (13)) and eqn (25) yields with (27) with the Lagrange equation the expression

$$\ddot{W}_{mn} + \left[\omega_{mn}^{2(M)} - \frac{g\rho}{\mu}\right]W_{mn} = \frac{4i\rho\omega}{\mu\pi^2}\sum_{\substack{k=0\\k\neq m}}^{\infty}\sum_{\substack{l=0\\l\neq n}}^{\infty} e^{i\omega t}A_{kl}\frac{mn[1-(-1)^{k+m}][1-(-1)^{n+l}]}{(m^2-k^2)(n^2-l^2)} \tag{30}$$

where $\omega_{mn}^{(M)}$ is the natural circular frequency of the membrane. It is

$$\omega_{mn}^{2(M)} = \frac{\pi^2 T_0}{\mu}\left[\frac{m^2}{a^2}+\frac{n^2}{b^2}\right]. \tag{31}$$

For this reason the functions $\cos(k\pi x/a)$ and $\cos(l\pi y/b)$ have been expanded into Fourier-sine-series with the coefficients

$$\alpha_m^{(k)} = \frac{2}{\pi}\frac{m[1-(-1)^{m+k}]}{(m^2-k^2)},\quad \cos\frac{k\pi x}{a} = \sum_{m=1}^{\infty}\alpha_m^{(k)}\sin\left(\frac{m\pi x}{a}\right)$$

$$\alpha_k^{(k)} = 0$$

and

$$\beta_n^{(l)} = \frac{2}{\pi}\frac{n[1-(-1)^{n+l}]}{[n^2-l^2]},\quad \cos\frac{l\pi y}{b} = \sum_{n=1}^{\infty}\beta_n^{(l)}\sin\left(\frac{n\pi y}{b}\right)$$

$$\beta_l^{(l)} = 0.$$

It may be noticed that values for either $k = 0$ or $l = 0$ and all other values except $k + m$ and $l + n$ being odd $\alpha_m^{(k)}$ and $\beta_n^{(l)}$ vanish.

The solution of the differential eqn (30)

$$W_{mn} = B_{mn}\,e^{i\omega t} \tag{32}$$

yields

$$B_{mn}\left[\omega_{mn}^{2(M)} - \omega^2 - \frac{\rho g}{\mu}\right] - \frac{4i\rho\omega}{\pi^2\mu}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}A_{kl}\frac{mn[1-(-1)^{k+m}][1-(-1)^{n+l}]}{(m^2-k^2)(n^2-l^2)} = 0 \tag{33}$$

while the compatibility equation $\partial\Phi/\partial z = \partial w/\partial t$ at $z = 0$ results with eqn (25) and (32) in

$$i\omega B_{mn} = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{4A_{kl}\omega_{kl}^{2(s)}}{g\pi^2}\left[\frac{\omega^2 - \dfrac{\omega_{kl}^{2(s)}}{\tanh^2(\pi h/ab)\sqrt{l^2a^2+k^2b^2}}}{\omega_{kl}^{2(s)}-\omega^2}\right]\frac{mn}{(m^2-k^2)(n^2-l^2)}\cdot$$
$$\cdot[1-(-1)^{m+k}][1-(-1)^{n+l}]. \tag{34}$$

The equations (33) and (34) represent ∞^2 homogeneous algebraic equations, of which the coefficient determinant yields the frequency equation for the determination of the coupled frequencies ω_{mn}. With the above remarks we notice that to $B_{2m-1,2n-1}$ only $A_{2k,2l}$ values appear, that to B_{2m2n} only $A_{2k-1\,2l-1}$, to $B_{2m\,2n-1}$ only $A_{2k-1\,2l}$ and that to $B_{2m-1\,2n}$ only $A_{2k\,2l-1}$ appear.

Multiplication with $i\omega$ of the first equation results in the possibility to eliminate the B_{mn} values. It is therefore:

$$\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{mn\,A_{kl}[1-(-1)^{m+k}][1-(-1)^{n+l}]}{(m^2-k^2)(n^2-l^2)}\left\{\frac{\rho\omega^2}{\mu}+\left[\omega_{mn}^{(M)2}-\omega^2-\frac{\rho g}{\mu}\right]\cdot\frac{\omega_{mn}^{(s)2}}{g}\right.$$
$$\left.\cdot\left[\frac{\omega^2-\omega_{kl}^{(s)2}/\tanh^2\left[\frac{\pi h}{ab}\sqrt{l^2a^2+k^2b^2}\right]}{(\omega_{mn}^{(s)2}-\omega^2)}\right]\right\}=0 \tag{35}$$

of which the coefficient determinant represents the frequency equation for the determination of the coupled frequencies ω. Truncating this infinite determinant for given m- and n-values to a finite number of rows and columns yields the approximate coupled frequencies of the system. The higher the order of the evaluated determinant, the better shall be the approximation of the lower frequency results.

In the case the bottom is described by an elastic *simply-supported plate*, the analysis of the solution is quite similar to the membrane case. The potential energy (29) has to be substituted by

$$V=\frac{D_p\pi^4}{8}ab\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\frac{m^2}{a^2}+\frac{n^2}{b^2}\right]^2W_{mn}^2 \tag{36}$$

which results with μ being substituted by $\rho_p\bar{h}_p$ in the same set of equation except for the fact that $\omega_{mn}^{(M)2}$ has to be substituted by $\omega_{mn}^{(P)2}$, the uncoupled natural frequencies of the elastic plate. It is:

$$\omega_{mn}^{(P)2}=\frac{D_p\pi^4}{\rho_p h_p}\left[\frac{m^2}{a^2}+\frac{n^2}{b^2}\right]^2. \tag{37}$$

Thus solving the above determinant of eqns (35) with $\omega_{mn}^{(P)2}$ instead of $\omega_{mn}^{(M)2}$ yields the coupled frequencies for the liquid-platebottom-system.

3.2 *Coverage of the liquid surface with an elastic member*

We distinguish here again two cases, namely the coverage of the free liquid surface with a flexible membrane and the case, where the surface is covered by an elastic plate. This procedure will result in an increase of the coupled frequencies of the liquid-structure-system. In this case we have to solve simultaneously the eqns (1)–(3) together with (21), (23) and (24) or (22), (23) and (24). If one uses a *flexible membrane* as a means of covering the free liquid surface, a velocity potential satisfying the Laplace equation (1) together with the rigid tank wall boundary conditions (2), (3) and (23) is given by

$$\Phi(x,y,z,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\bar{A}_{mn}\frac{\cosh\left[\frac{\pi}{ab}\sqrt{n^2a^2+m^2b^2}\,(z+h)\right]}{\cosh\left[\frac{\pi h}{ab}\sqrt{n^2a^2+m^2b^2}\right]}\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{i\omega t}. \tag{38}$$

Here the rigid bottom of the container is taken at $z=-h$, while the flexible membrane is located at $z=0$. The values $\bar{A}_{mn}$ are integration constants and ω are the coupled frequencies of the liquid-membrane-system, which have to be determined. The procedure of the solution continues as in the previous section. With the loading of the membrane by the liquid motion

$$P(x,y,t)=-i\rho\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\omega_{kl}\bar{A}_{kl}\cos\left(\frac{k\pi x}{a}\right)\cos\left(\frac{l\pi y}{b}\right)e^{i\omega t}-\rho g w \tag{39}$$

the equation for the amplitude of the membrane yields

$$\ddot{W}_{mn} + \left[\omega_{mn}^{2(M)} + \frac{\rho g}{\mu}\right] W_{mn} = -\frac{4i\rho\omega}{\mu\pi^2} \sum_{\substack{k=1 \\ k\neq m}}^{\infty} \sum_{\substack{l=1 \\ l\neq n}}^{\infty} e^{i\omega t} \bar{A}_{kl} \frac{mn[1-(-1)^{k+m}][1-(-1)^{n+l}]}{(m^2-k^2)(n^2-l^2)}. \tag{40}$$

The solution of this differential equation is of the form of eqn (32) and results with the compatibility condition (24) in

$$\bar{B}_{mn}\left[\omega_{mn}^{2(M)} - \omega^2 + \frac{\rho g}{\mu}\right] + \frac{4i\rho\omega}{\mu\pi^2} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \bar{A}_{kl} \frac{mn[1-(-1)^{k+m}][1-(-1)^{n+l}]}{(m^2-k^2)(n^2-l^2)} = 0 \tag{41}$$

and

$$i\omega\bar{B}_{mn} = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{4\bar{A}_{kl}\omega_{kl}^{2(s)}}{g\pi^2} \frac{mn[1-(-1)^{k+m}][1-(-1)^{l+n}]}{(m^2-k^2)(n^2-l^2)}. \tag{42}$$

These eqns (41) and (42) represent ∞^2 homogeneous algebraic equations, of which the coefficient determinant yields the frequency equation for the coupled frequencies of the membrane-liquid system. The elimination of the integration constants $\bar{B}_{mn}$ yields:

$$\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{mn\bar{A}_{kl}[1-(-1)^{m+k}][1-(-1)^{n+l}]}{(m^2-k^2)(n^2-l^2)} \left\{\frac{\omega_{kl}^{2(s)}}{g}\left[\omega_{mn}^{2(M)} - \omega^2 + \frac{\rho g}{\mu}\right] - \frac{\rho\omega^2}{\mu}\right\} = 0$$

$$m, n, = 1, 2, 3, \ldots \tag{43}$$

of which the coefficient determinant represents the frequency equation for the determination of the coupled frequencies ω. Truncating this infinite determinant for given m- and n-values to a finite number of rows and columns results in approximate coupled lower frequencies.

In the case of a coverage by an *elastic plate*, which is considered to be simply-supported, the analysis for the determination of the coupled frequencies of the plate-liquid-system is quite similar as in the membrane case and results in the same set of equations except for the fact, that μ has to be substituted by $\rho_p \bar{h}_p$ and $\omega_{mn}^{2(M)}$ has to be replaced by $\omega_{mn}^{2(P)}$.

Numerical evaluation. The coupled frequencies have been numerically evaluated for a rectangular container of the side ratio $b/a = 0.5$ and the parameter $\rho a/\mu = 1000$. The lower coupled frequencies are presented together with the uncoupled frequencies $\omega^2/(g/a)$ as functions of the tension variable $T_0/\mu g a$ in the case of a membrane and as functions of the stiffness parameter $D/\mu g a^3$ in the case of a plate. Two typical liquid heights have been investigated. The first one exhibits a rather highly filled container of the liquid height ratio $h/a = 1.0$, while the second case considers a quite low filling ratio $h/a = 0.1$. Figures 2 exhibit the frequencies of the flexible membrane-bottom-liquid system with a free liquid surface. The lower coupled and uncoupled frequencies are presented for those values of m and n which modes interact with each other. The uncoupled frequencies of liquid and membrane are for $hj/a = 0.1$ (---) and for $h/a = 1.0$ (——) presented. The coupled membrane and liquid frequencies are exhibited for $h/a = 0.1$ (-·-) and for $h/a = 1.0$ as a full line (——). It may be noticed that for a liquid height ratio $h/a = 1.0$ coupled liquid frequency slightly shifts in magnitude below the value of the uncoupled frequencies, while the coupled membrane frequencies exhibit a considerable decrease in comparison with the uncoupled frequencies of the membrane. The effect of interaction of the flexible membrane bottom with the liquid surface is, of course, much more pronounced in the case of a low liquid height, as is shown for $h/a = 0.1$ in Fig. 2. The coupled liquid frequencies shift to much higher values and exhibit with decreasing tension an increase of magnitude, while the coupled membrane frequencies are of much lower magnitude than the uncoupled membrane frequencies and show with decreasing tension a more rapid decrease in magnitude. Similar behavior may be observed for a container with a thin elastic plate as its

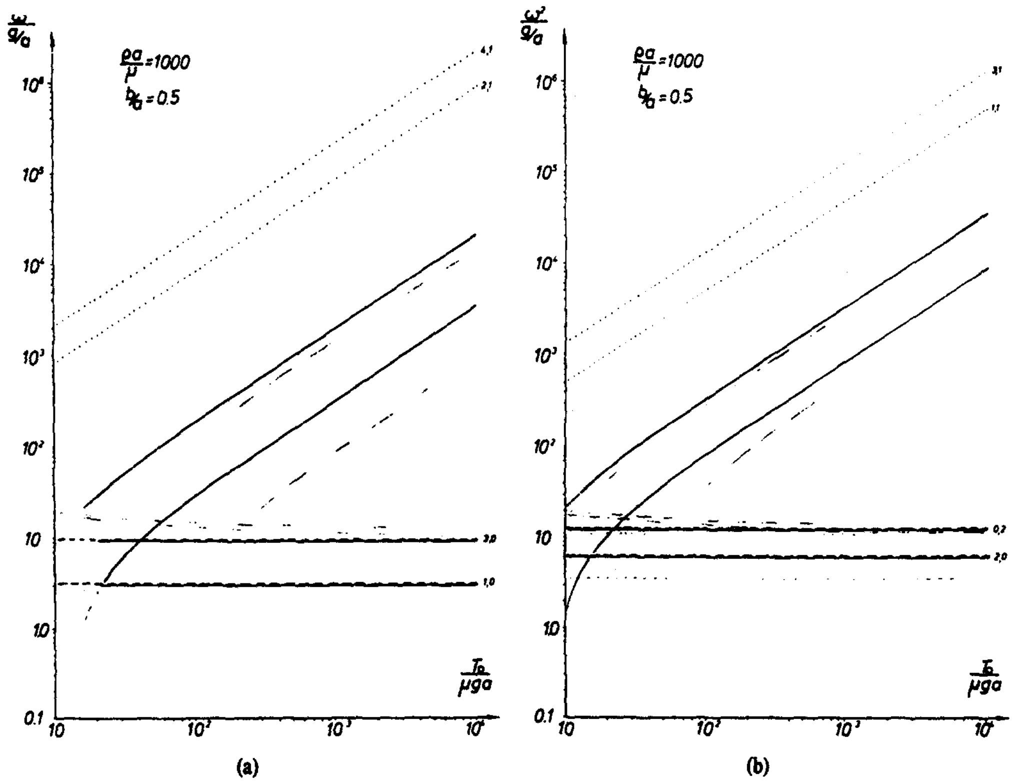

Fig. 2. Coupled frequencies of flexible membrane-bottom-liquid system {—— coupled frequencies, --- uncoupled frequencies} for liquid height $h/a = 1.0$; {-·- coupled frequencies, ... uncoupled frequencies} for liquid height $h/a = 0.1$.

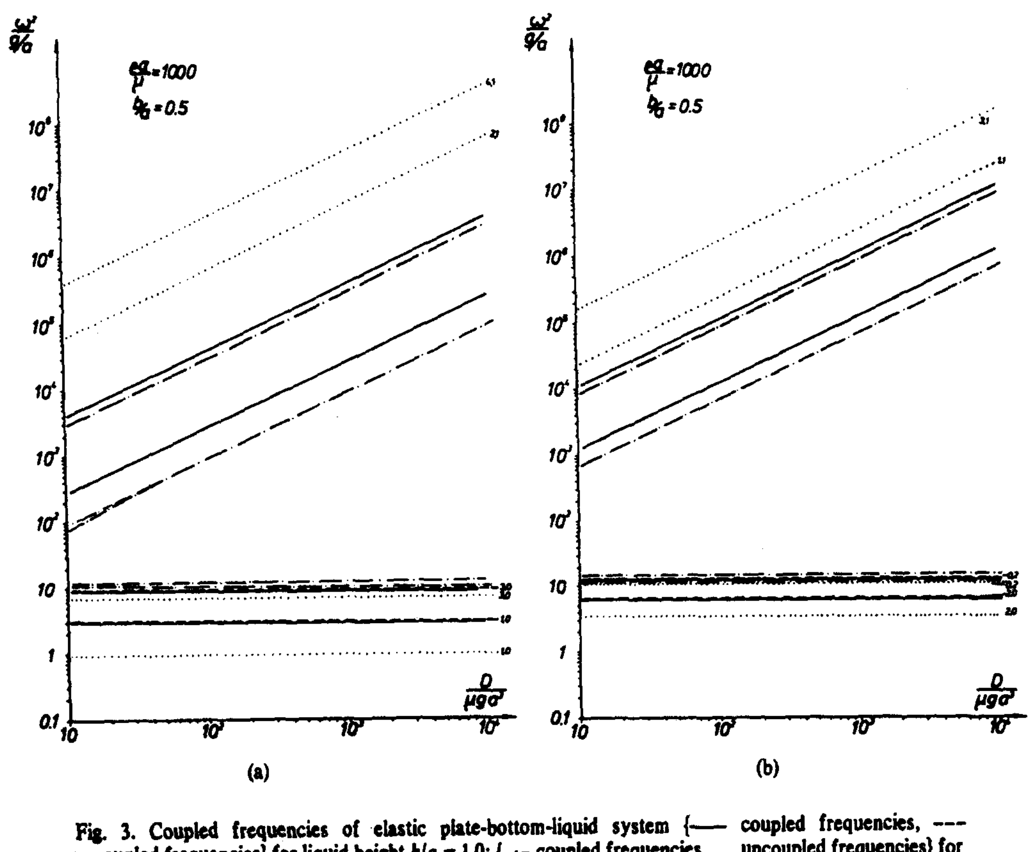

Fig. 3. Coupled frequencies of elastic plate-bottom-liquid system {—— coupled frequencies, --- uncoupled frequencies} for liquid height $h/a = 1.0$; {-·- coupled frequencies, ... uncoupled frequencies} for liquid height $h/a = 0.1$.

bottom and a free liquid surface. It is shown in Fig. 3. To investigate the influence of varying liquid height and show the strong interaction of bottom and free liquid surface for the lower liquid heights, the coupled and uncoupled frequencies have been presented in the Figs 4. In these figures the uncoupled liquid frequencies presented by the dashed line (---), the uncoupled frequencies of the membrane by (——) and those of the plate by the dotted lines (...). The coupled frequencies are shown as dash-dotted (-·-) lines and full lines (——). The uncoupled liquid frequencies approach, of course, with decreasing liquid height h the frequency zero, while uncoupled membrane and plate frequencies are constant values. The coupled liquid frequencies increase with decreasing liquid height, while the coupled membrane- and plate frequencies are much reduced and decrease for decreasing low liquid heights. The upper two full lines represent the coupled membrane frequencies which correspond to the dashed (---) uncoupled frequencies, while the upper (-·-) lines are coupled plate frequencies and correspond to the dotted lines (...) for the uncoupled frequencies of the plate.

If a container is completely rigid and its free liquid surface is covered with a flexible membrane or an elastic plate the usually low free liquid surface frequencies may be shifted by this method to much higher values. This may be a desirable effect in space vehicles and space laboratories in order to shift the natural frequencies away from control frequencies, thus getting rid of a strong interaction of the liquid and control system and a low-value jitter, which definitely is undesirable for experiments to be performed on a space lab. In Figs. 5 the effect of a membrane cover on the liquid surface is exhibited besides the uncoupled frequencies of the free liquid surface and membrane alone. The liquid height ratios $h/a = 0.1$ (...) and $h/a = 1.0$ (---) are exhibited with the uncoupled membrane frequencies as dotted lines (...), while the coupled frequencies are presented for $h/a = 0.1$ as (-·-) and for $h/a = 1.0$ as a full (——) line. It may be mentioned that always two uncoupled frequencies are shown while for the coupled frequencies the four lowest ones are presented. For $h/a = 0.1$ the coupled frequencies exhibit lower values as those for $h/a = 1.0$ With decreasing membrane tension the coupled frequencies approach those of the case of the free liquid surface. Similar results are obtained for the covering of the free liquid surface with a thin elastic plate. They are exhibited in Fig. 6. The strong influence on the coupling of the system by the liquid height is presented in Figs. 7, where the uncoupled frequencies of the liquid (---), the plate (...), the membrane (———) and the coupled frequencies of plate-liquid height ratio h/a.

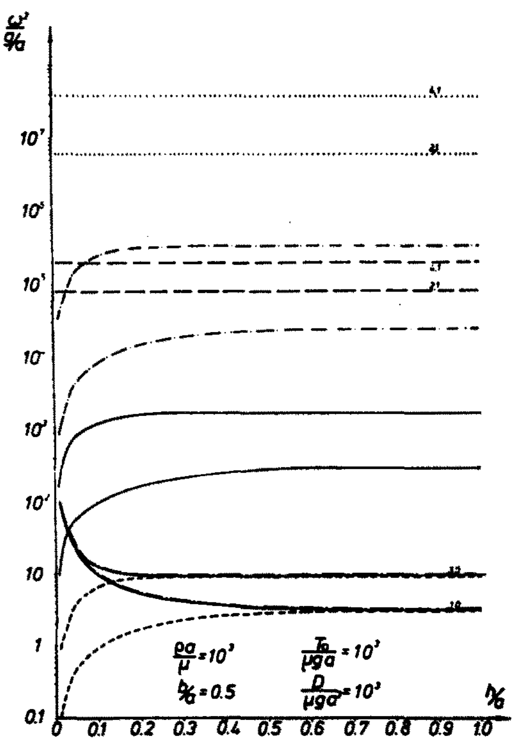

Fig. 4. Influence of liquid height upon the coupled liquid frequencies.

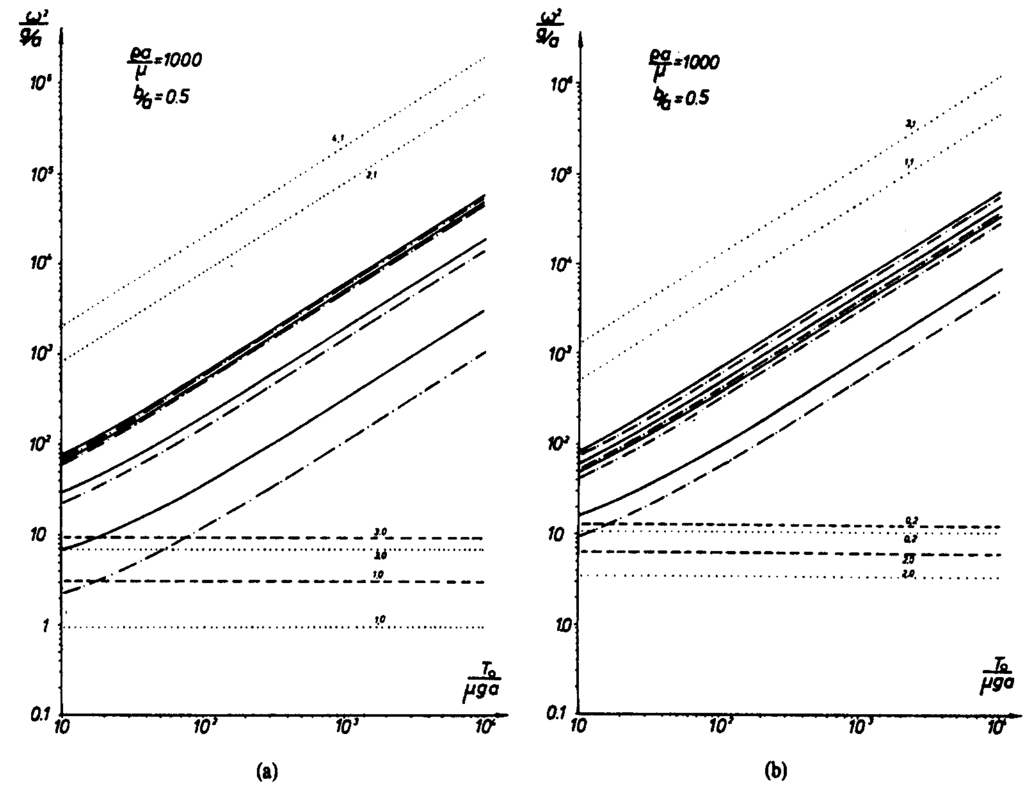

Fig. 5. Coupled frequencies of flexible membrane-cover-liquid system {—— coupled frequencies, --- uncoupled frequencies} for liquid height $h/a = 1.0$, {-·- coupled frequencies, ... uncoupled frequencies} for liquid height $h/a = 0.1$.

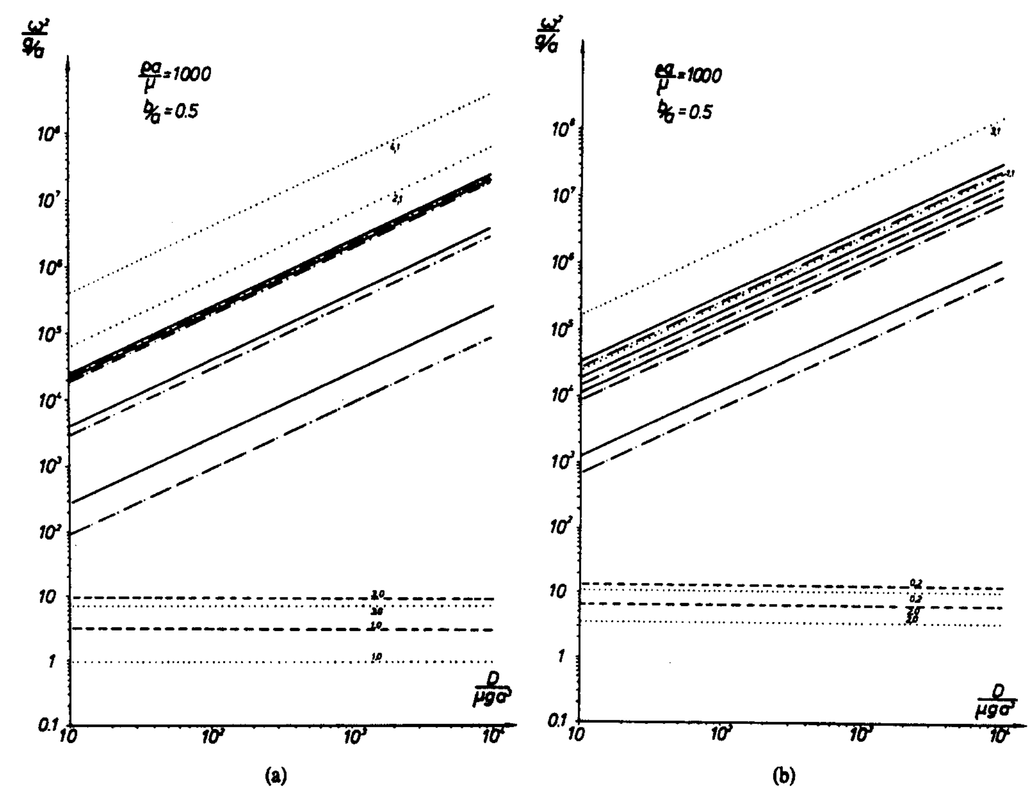

Fig. 6. Coupled frequencies of elastic plate-cover-liquid system {—— coupled frequencies, --- uncoupled frequencies} for liquid height $h/a = 1.0$; {-·- coupled frequencies, ... uncoupled frequencies} for liquid height $h/a = 0.1$.

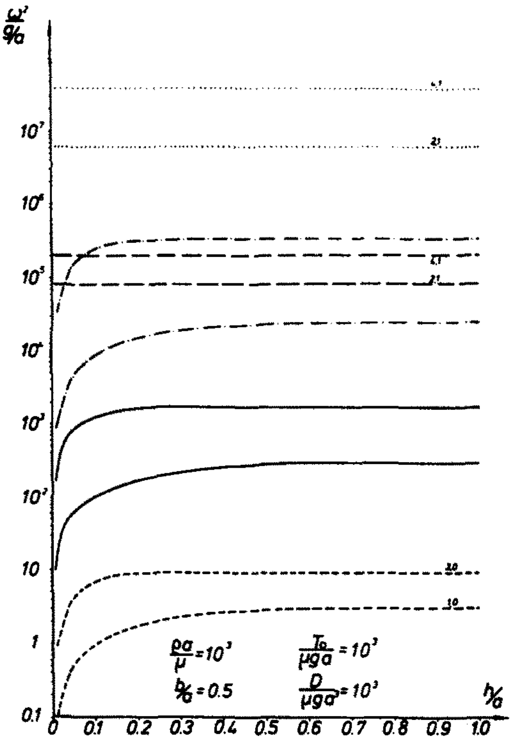

Fig. 7. Influence of liquid-height upon the coupled frequencies of the liquid surface covered system (--- liquid uncoupled, ... plate uncoupled, --- membrane uncoupled; -·-·-·- coupled liquid-plate system, —— coupled liquid-membrane system).

Only for low liquid heights there is a further strong decrease in the other-wise already reduced magnitude of the coupled frequencies compared with the uncoupled ones.

The numerical results have been obtained by limiting the magnitude of the summation indices k, l, m and n. If for reasons of comparison the maximum of those values was changed from 8 to 10, the difference in the results showed in the first and second coupled frequencies in all cases a change of less than 3%.

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